Sample Question Paper - 9
Mathematics (041)
Class- XII, Session: 2021-22
TERM II

Time Allowed: 2 hours **Maximum Marks: 40**

General Instructions:

1. This question paper contains three sections – A, B and C. Each part is compulsory.
2. Section - A has 6 short answer type (SA1) questions of 2 marks each.
3. Section – B has 4 short answer type (SA2) questions of 3 marks each.
4. Section - C has 4 long answer-type questions (LA) of 4 marks each.
5. There is an internal choice in some of the questions.
6. Q 14 is a case-based problem having 2 sub-parts of 2 marks each.

Section A

1. Evaluate: $\int \frac{\sin 2x}{a\cos^2 x + b\sin^2 x} dx$ **[2]**

OR

Evaluate: $\int \sqrt{\tan\theta} d\theta$

2. Solve $\frac{dy}{dx} + (\sec x)y = \tan x$. **[2]**
3. If the points A (m, - 1), B(2, 1) and C(4,5) are collinear, find the value of m. **[2]**
4. Show that the points (1, 1, 1) and (-3, 0,1) are equidistant from the plane 3x + 4y - 12z + 13 = 0. **[2]**
5. The probabilities of A, B and C solving a problem are $\frac{1}{3}$, $\frac{2}{7}$ and $\frac{3}{8}$ respectively. If all three try to solve the problem simultaneously, find the probability that exactly one of them can solve it. **[2]**
6. Let A and B be the events such that P(A) = $\frac{7}{13}$, P(B) = $\frac{9}{13}$ and P(A∩B) = $\frac{4}{13}$ find $P(\bar{B}/\bar{A})$ **[2]**

Section B

7. Evaluate the integral: $\int_0^1 \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)dx$ **[3]**
8. Find the particular solution of the differential equation $\left(1-x^2\right)\frac{dy}{dx} - xy = x^2$, given that y = 2 when x = 0. **[3]**

OR

Solve the initial value problem: $x\frac{dy}{dx}$ + y = x cos x + sin x, y$\left(\frac{\pi}{2}\right)$ = 1

9. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that each one is perpendicular to the vector obtained by sum of the other two and $|\vec{a}| = 3, |\vec{b}| = 4$ and $|\vec{c}| = 5$, then prove that $|\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$. **[3]**
10. Find the vector and cartesian equations of the line through the point (1, 2, - 4) and perpendicular to the two lines $\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$ and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$. **[3]**

OR

Find the equation of the plane determined by the intersection of the lines $\frac{x+3}{3} = \frac{y}{-2} = \frac{z-7}{6}$ and

$\frac{x+6}{1}=\frac{y+5}{-3}=\frac{z-1}{2}$.

Section C

11. Evaluate $\int(\sqrt{\cot x}+\sqrt{\tan x})$dx. **[4]**

12. Sketch the graph of y = Ix + 3I and evaluate the area under the curve y = Ix + 3I above X - axis and between x = - 6 to x = 0. **[4]**

OR

Find the area bounded by the curve y = 4 - x^2 and the lines y = 0, y = 3.

13. Prove that if a plane has the intercepts a,b,c is at a distance of p units from the origin then $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}$ **[4]**

CASE-BASED/DATA-BASED

14. Three bags contain a number of red and white balls as follows: **[4]**

Bag 1 : 3 red balls, Bag 2 : 2 red balls and 1 white ball and Bag 3 : 3 white balls.

The probability that bag a will be chosen and a ball is selected from it is $\frac{1}{6}$. What is the probability that

i. a red ball will be selected?

ii. a white ball is selected?

Solution

MATHEMATICS 041

Class 12 - Mathematics

Section A

1. Let I = $\int \frac{\sin 2x}{a\cos^2 x+b\sin^2 x}dx$ ...(i)

   Let a $\cos^2$ x + b $\sin^2$ x = t then,

   d(a $\cos^2$ x + b $\sin^2$ x = dt

   [a(2 cos x(-sin x))+b (2 sin x cos x)] dx = dt

   $\Rightarrow$ [-a(2 sin x cos x) + b(2 sin x cos x)] dx = dt

   $\Rightarrow$[-a sin 2x + b sin 2x] dx = dt

   $\Rightarrow$ sin 2x(b - a) dx = dt

   $\Rightarrow$ dx= $\frac{dt}{(b-a)sin2x}$

   Putting a $\cos^2$ x + b $\sin^2$ x= and dx = $\frac{dt}{(b-a)sin2x}$ in equation (i), we get

   $I=\int \frac{\sin 2x}{t}\times\frac{dt}{(b-a)\sin 2x}$

   $=\frac{1}{b-a}\int\frac{dt}{t}$

   $=\frac{1}{b-a}\log|t|+c$

   = $\frac{1}{b-a}$ log |a $\cos^2$ x + b $\sin^2$ x| + c [a $\cos^2$ x + b $\sin^2$ x = t]

   OR

   Let, I = $\int\sqrt{\tan\theta}d\theta$

   Now let tanθ = x^2. Then, we have

   d(tanθ) = d(x^2) $\Rightarrow$ $\sec^2\theta$ dθ = 2xdx

   $\Rightarrow$ dθ = $\frac{2xdx}{\sec^2\theta}$ = $\frac{2xdx}{1+\tan^2\theta}$ = $\frac{2xdx}{1+x^4}$

   I = $\int\sqrt{x^2}\cdot\frac{2xdx}{1+x^4}$ = $\int\frac{2x^2}{x^4+1}$ dx = $\int\frac{2}{x^2+1/x^2}$ dx = $\int\frac{1+1/x^2+1-1/x^2}{x^2+1/x^2}$dx

   $\Rightarrow$ I = $\int\frac{1+1/x^2}{x^2+1/x^2}$ dx + $\int\frac{1-1/x^2}{x^2+1/x^2}$ dx

   $\Rightarrow$ I = $\int\frac{1+1/x^2}{(x-1/x)^2+2}$dx + $\int\frac{1-1/x^2}{(x+1/x)^2-2}dx$

   Putting x - $\frac{1}{x}$ = u in first integral and x + $\frac{1}{x}$ = v in second integral, we get

   I = $\int\frac{du}{u^2+(\sqrt{2})^2}$ + $\int\frac{dv}{v^2-(\sqrt{2})^2}$

   $\Rightarrow$ I = $\frac{1}{\sqrt{2}}$ $\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)$ + $\frac{1}{2\sqrt{2}}\log\left|\frac{v-\sqrt{2}}{v+\sqrt{2}}\right|$+ C

   $\Rightarrow$ I = $\frac{1}{\sqrt{2}}$ $\tan^{-1}\left(\frac{x-1/x}{\sqrt{2}}\right)$ + $\frac{1}{2\sqrt{2}}\log\left|\frac{x+1/x-\sqrt{2}}{x+1/x+\sqrt{2}}\right|$ + C

2. The given equation is of the form

   $\frac{dy}{dx}+Py=Q$, where P = sec x and Q = tan x.

   Thus, the given equation is linear.

   $\text{IF}=e^{\int P\,dx}=e^{\int\sec x\,dx}=e^{\log(\sec x+\tan x)}=(\sec x+\tan x)$

   So, the required solution is

   $y\times\text{IF}=\int\{Q\times(\text{IF})\}dx+C,$

   i.e., $y(\sec x+\tan x)=\int\tan x(\sec x+\tan x)dx+C$

   $=\int\sec x\tan x\,dx+\int\tan^2 x dx+c$

   $=\sec x+\int\left(\sec^2 x-1\right)dx+C$

   = sec x + tan x - x + C.

   Hence, y(sec x + tan x) = sec x + tan x - x + C is the required solution.

3. The given points are A(m, -1), B(2, 1) and C(4, 5)

   Now,we have

$\overrightarrow{AB} = (2\hat{i} + \hat{j}) - (m\hat{i} - \hat{j}) = (2 - m)\hat{i} + 2\hat{j}$
$\overrightarrow{AC} = (4\hat{i} + 5\hat{j}) - (m\hat{i} - \hat{j}) = (4 - m)\hat{i} + 6\hat{j}$
If A, B, C are collinear, then
$\overrightarrow{AB} = \lambda\overrightarrow{AC}$
$\Rightarrow (2 - m)\hat{i} + 2\hat{j} = \lambda[(4 - m)\hat{i} + 6\hat{j}]$
$\Rightarrow 2 - m = \lambda(4 - m)$ and $2 = 6\lambda$
$\Rightarrow \lambda = \frac{1}{3}$
and 2 - m $= \frac{1}{3}(4 - m)$
$\Rightarrow$ 6 - 3m = 4 - m
$\Rightarrow$ 2m = 2
$\Rightarrow$ m = 1
Therefore, the value of m is 1.

4. Given:
Points: A(1, 1, 1) and B(–3, 0, 1)
Plane: π = 3x + 4y – 12z + 13 = 0
We know, the distance of point (x_1,y_1,z_1) from the plane
π: ax + by + cz + d = 0: ax + by + cz + d = 0 is given by:
$p = \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$
$\Rightarrow$ Distance of (1, 1, 1) from the plane = $\left|\frac{(3)(1)+(4)(1)+(-12)(1)+13}{\sqrt{3^2+4^2+(-12)^2}}\right|$
= $\frac{8}{13}$ units
$\Longrightarrow$ Distance of (–3, 0, 1) from the plane = $\left|\frac{(3)(-3)+(4)(0)+(-12)(1)+13}{\sqrt{3^2+4^2+(-12)^2}}\right|$
= $\frac{8}{13}$ units
$\therefore$ the points (1, 1, 1) and (–3, 0, 1) are equidistant from the plane
3x + 4y – 12z + 13 = 0.

5. Let E_1, E_2 and E_3 be the events that the problem is solved by A, B and C respectively. Therefore,we have,
$P(E_1) = \frac{1}{3}$, $P(E_2) = \frac{2}{7}$ and $P(E_3) = \frac{3}{8}$
Exactly one of A, B and C can solve the problem in the following mutually exclusive ways:
   i. A solves but B and C do not solve i.e. $E_1 \cap \bar{E}_2 \cap \bar{E}_3$
   ii. B solves but A and C do not solve i.e. $\bar{E}_1 \cap E_2 \cap \bar{E}_3$
   iii. C solves but A and B do not solve i.e. $\bar{E}_1 \cap \bar{E}_2 \cap E_3$
Therefore, Required probability = P (I or II or III)
= $P\left[\left(E_1 \cap \bar{E}_2 \cap \bar{E}_3\right) \cup \left(\bar{E}_1 \cap E_2 \cap \bar{E}_3\right) \cup \left(\bar{E}_1 \cap \bar{E}_2 \cap E_3\right]\right.$
= $P\left(E_1 \cap \bar{E}_2 \cap \bar{E}_3\right) + \left(\bar{E}_1 \cap E_2 \cap \bar{E}_3\right) \cup \left(\bar{E}_1 \cap \bar{E}_2 \cap E_3\right]$
= $P(E_1)\,P\left(\bar{E}_2\right)P\left(\bar{E}_3\right) + P\left(\bar{E}_1\right)P(E_2)P\left(\bar{E}_3\right) + P\left(\bar{E}_1\right)P\left(\bar{E}_2\right)P(E_3)$
= $\frac{1}{3}\left(1 - \frac{2}{7}\right)\left(1 - \frac{3}{8}\right) + \left(1 - \frac{1}{3}\right)\left(\frac{2}{7}\right)\left(1 - \frac{3}{8}\right)$ + $\left(1 - \frac{1}{3}\right)\left(1 - \frac{2}{7}\right)\left(\frac{3}{8}\right)$
= $\frac{1}{3} \times \frac{5}{7} \times \frac{5}{8} + \frac{2}{3} \times \frac{2}{7} \times \frac{5}{8} + \frac{2}{3} \times \frac{5}{7} \times \frac{3}{8}$
= $\frac{25}{168} + \frac{5}{42} + \frac{5}{28} = \frac{25}{56}$

6. $P(\bar{B}/\bar{A}) = \frac{P(\bar{A}\cap\bar{B})}{P(\bar{A})}$
Now, by De-Morgan's Law, $(A \cup B)^C = A^C \cap B^C$
$\therefore P(\bar{A} \cap \bar{B}) = P(A \cup B)^c$
Therefore,we have,

$P(\frac{\bar{B}}{\bar{A}})$
$= \frac{P(\bar{A} \cap \bar{B})}{P(\bar{A})}$
$= \frac{P(A \cup B)^c}{P(\bar{A})}$
$= \frac{1 - P(A \cup B)}{1 - P(A)}$
$= \frac{1 - \frac{12}{13}}{1 - \frac{7}{13}}$
$= \frac{1}{6}$

Section B

7. To solve this we will use substitution.
   Let x = tanθ
   dx = sec$^2\theta$ dθ
   Now, x = 0 $\Rightarrow$ θ = 0
   x = 1 $\Rightarrow$ $\theta = \frac{\pi}{4}$
   $\int\limits_0^1 \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx$
   $= \int\limits_0^{\frac{\pi}{4}} \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) \sec^2\theta d\theta$
   $= \int\limits_0^{\frac{\pi}{4}} \cos^{-1}(\cos 2\theta) \sec^2\theta d\theta \left[\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$
   $= \int\limits_0^{\frac{\pi}{4}} 2\theta \sec^2\theta d\theta$
   Using by parts, we get
   $\int 2\theta \sec^2\theta d\theta$
   $= 2\left[\theta \int \sec^2\theta d\theta - \int \left(\int \sec^2\theta d\theta\right) \frac{d\theta}{d\theta} \times d\theta\right]$
   $= 2\left[\theta \tan\theta - \int \tan\theta d\theta\right]$
   $\therefore \int\limits_0^{\frac{\pi}{4}} 2\theta \sec^2\theta d\theta$
   $= 2[\theta \tan\theta + \log\cos\theta]_0^{\frac{\pi}{4}}$ $[\because \int \tan\theta d\theta = \log\cos\theta]$
   $= 2\left[\left(\frac{\pi}{4}\tan\frac{\pi}{4} + \log\cos\frac{\pi}{4}\right) - (0 \times \tan 0 + \log\cos 0)\right]$
   $= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - 0 - 0\right]$
   $= 2\left(\frac{\pi}{4} + \log\frac{1}{\sqrt{2}}\right)$
   $= \frac{\pi}{2} - \log 2$
   $\therefore \int\limits_0^1 \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) dx = \frac{\pi}{2} - \log 2$
8. The given differential equation may be rewritten as,
   $\frac{dy}{dx} - \frac{x}{(1-x^2)} \cdot y = \frac{x^2}{(1-x^2)}$. ...(i)
   This is of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{-x}{(1-x^2)}$ and $Q = \frac{x^2}{(1-x^2)}$.
   Thus, the given differential equation is linear.
   Therefore,we have,
   $\text{IF} = e^{\int P\,dx} = e^{\int \frac{-x}{(1-x^2)} dx} = e^{\frac{1}{2}\int \frac{-2x}{(1-x^2)} dx} = e^{\frac{1}{2}\log(1-x^2)}$
   $= e^{\log\sqrt{1-x^2}} = \sqrt{1-x^2}$
   Therefore, required solution is given by,

$y \times IF = \int (Q \times IF) dx + C$

i.e., $y \times \sqrt{1-x^2} = \int \frac{x^2}{(1-x^2)} \times \sqrt{1-x^2}\, dx + C$

$= \int \frac{x^2}{\sqrt{1-x^2}} dx + C$

$= \int \frac{\{-(1-x^2)+1\}}{\sqrt{1-x^2}} dx + C$

$= -\int \sqrt{1-x^2}\, dx + \int \frac{1}{\sqrt{1-x^2}} dx + C$

$= -\left\{\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x\right\} + \sin^{-1} x + C$

$= \frac{-x\sqrt{1-x^2}}{2} + \frac{1}{2}\sin^{-1} x + C$

$\therefore \quad y = \frac{-x}{2} + \frac{\sin^{-1} x}{2\sqrt{1-x^2}} + \frac{C}{\sqrt{1-x^2}}$ ...(ii)

It is being given that when x = 0, then y = 2.

Put x = 0 and y = 2 in (ii), we get C = 2.

Hence, $y = \frac{-x}{2} + \frac{\sin^{-1} x}{2\sqrt{1-x^2}} + \frac{2}{\sqrt{1-x^2}}$ is the required solution.

OR

The given differential equation is,

$x\frac{dy}{dx}$ + y = x cos x + sin x

$\frac{dy}{dx} + \frac{y}{x}$ = cos x + $\frac{\sin x}{x}$

It is a linear differential equation. Comparing it with,

$\frac{dy}{dx}$ + Py = Q

P = $\frac{1}{x}$, Q = cos x + $\frac{\sin x}{x}$

I.F. $= e^{\int p dx}$

$= e^{\int \frac{1}{x} dx}$

= $e^{\log |x|}$, x > 0

= x, x > 0

Solution of the equation is given by,

y × (I.F.) = $\int$ Q × (I.F.) dx + c

y(x) = $\int \left(\cos x + \frac{\sin x}{x}\right)$ x dx + c

= $\int$ (x cos x + sin x) dx + c

xy = $\int$ x cos x dx + $\int$ sin x dx + c

= x$\int$ cos x dx - $\int$ (1 × $\int$ cos x dx) dx - cos x + c

= x sin x - $\int$ sin x dx - cos x + c

= x sin x + cos x - cos x + c

xy = x sin x + c

Put x = $\frac{\pi}{2}$, y = 1

$\frac{\pi}{2} = \frac{\pi}{2}$ + c

c = 0

Put c = 0 in equation (i),

xy = x sin x

y = sin x

9. According to the question ,

$\vec{a} \perp (\vec{b}+\vec{c}), \vec{b} \perp (\vec{c}+\vec{a}), \vec{c} \perp (\vec{a}+\vec{b})$

and $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$

To prove $|\vec{a}+\vec{b}+\vec{c}| = 5\sqrt{2}$

Consider, $|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c}) \left[\because |\vec{x}|^2 = \vec{x}.\vec{x}\right]$

$= \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{a}\cdot\vec{c} + \vec{b}.\vec{a} + \vec{b}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} + \vec{c}\cdot\vec{b} + \vec{c}.\vec{c}$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + \vec{a}\cdot(\vec{b}+\vec{c}) + \vec{b}\cdot(\vec{a}+\vec{c}) + \vec{c}.(\vec{a}+\vec{c})$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 0 + 0 + 0$

$[\because \vec{a} \perp (\vec{b} + \vec{c})]$

$\therefore\ \vec{a}.(\vec{b} + \vec{c}) = 0$

Similarly, $\vec{b}.(\vec{a} + \vec{c}) = 0$

and $\vec{c}.(\vec{a} + \vec{b}) = 0$

= $3^2 + 4^2 + 5^2$ = 9 + 16 + 25

$\Rightarrow |\vec{a} + \vec{b} + \vec{c}|^2 = 50$

$\Rightarrow \quad |\vec{a} + \vec{b} + \vec{c}| = 5\sqrt{2}$

10. According to the question, the equations of lines are

$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$

$\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$

Comparing with vector form of equation of line $\vec{r} = \vec{a} + \lambda\vec{b}$, we get

$\Rightarrow \overrightarrow{b_1} = 3\hat{i} - 16\hat{j} + 7\hat{k}$

$\Rightarrow \overrightarrow{b_2} = 3\hat{i} + 8\hat{j} - 5\hat{k}$

$$\vec{b} = \overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$$

$= \hat{i}(80 - 56) - \hat{j}(-15 - 21) + \hat{k}(24 + 48)$

$= 24\hat{i} + 36\hat{j} + 72\hat{k} = 12(2\hat{i} + 3\hat{j} + 6\hat{k})$

The required line is perpendicular to the given lines.

So, it is parallel to $\vec{b}_1 \times \vec{b}_2$ Now, the equation of a line passing through the point $(1, 2, -4)$ and parallel to $24\hat{i} + 36\hat{j} + 72\hat{k}$ or $(2\hat{i} + 3\hat{j} + 6\hat{k})$ is

$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{j} + 3\hat{j} + 6\hat{k})$

which is required vector equation of a line.

For cartesian equation, put $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

$\Rightarrow x\hat{i} + y\hat{j} + z\hat{k} = (1 + 2\lambda)\hat{i} + (2 + 3\lambda)\hat{j} + (-4 + 6\lambda)\hat{k}$

Comparing the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$,

$\Rightarrow x = 1 + 2\lambda, y = 2 + 3\lambda$ and $z = -4 + 6\lambda$

$\Rightarrow \quad \frac{x-1}{2} = \lambda, \frac{y-2}{3} = \lambda$ and $\frac{z+4}{6} = \lambda$

$\therefore \quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$

which is the required cartesian equation of a line.

OR

The given equations of the lines are

$\frac{x+3}{3} = \frac{y}{-2} = \frac{z-7}{6}$ ...(1)

$\frac{x+6}{1} = \frac{y+5}{-3} = \frac{z-1}{2}$ ...(2)

Let the direction ratios of the plane be proportional to a, b, c.

since the plane contains the line (1), it should pass through (-3, 0, 7) and is parallel to the line (1).

Equation of the plane through (1) is

a(x + 3) + b(y) + c(z - 7) = 0 ...(3)

where 3a - 2b + 6c = 0 ...(4)

since the plane contains line (2), the plane is parallel to line (2) also.

$\Rightarrow$ a - 3b + 2c = 0 ...(5)

Solving (4) and (5) using cross-multiplication, we get

$\frac{a}{14} = \frac{b}{0} = \frac{c}{-7}$

Substituting, b and c in (3), we get

14(x + 3) + 0(y) - 7(z - 7) = 0

$\Rightarrow$ 2(x + 3) + 0(y) - 1(z - 7) = 0

$\Rightarrow$ 2x - z + 13 = 0.

Section C

11. Let $I = \int[\sqrt{\cot x} + \sqrt{\tan x}]dx$

$= \int \frac{1}{\sqrt{tanx}} + \sqrt{tanx}\,[\because cotx = \frac{1}{tanx}]$

$$= \int \sqrt{\tan x}\left[1 + \frac{1}{\left(\sqrt{\tan x}\right)^2}\right]dx$$

put $tanx = t^2 \implies sec^2xdx = 2tdt$

$\Rightarrow \quad dx = \frac{2t}{sec^2x}dt$

$\Rightarrow \quad dx = \frac{2t}{1+tan^2x}dt[\because 1 + \tan^2 x = \sec^2 x]$

$\Rightarrow \quad dx = \frac{2t}{1+(t^2)^2}[tanx = t^2]$

$\Rightarrow \quad dx = \frac{2t}{1+t^4}$

$\therefore \quad I = \int t\left(1 + \frac{1}{t^2}\right)\frac{2t}{(1+t^4)}dt[\because tanx = t^2 \implies \sqrt{tanx} = t]$

$= 2\int \frac{t^2+1}{t^4+1}dt$

On dividing numerator and denominator by t^2 , we get

$$I = 2\int \frac{\left(1+\frac{1}{t^2}\right)}{\left(t^2+\frac{1}{t^2}\right)}dt$$

$$= 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-2+2}dt$$

$$= 2\int \frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}dt$$

Again, put $t - \frac{1}{t} = y \Rightarrow \left(1 + \frac{1}{t^2}\right)dt = dy$

$\therefore I = 2\int \frac{dy}{y^2+(\sqrt{2})^2}$

$I = \frac{2}{\sqrt{2}}\tan^{-1}\frac{y}{\sqrt{2}} + C\left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c\right]$

$= \sqrt{2}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}} + C \quad \left[\text{put } y = t - \frac{1}{t}\right]$

$= \sqrt{2}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right) + C$

$= \sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right) + C\ [Put\ t^2 = tanx]$

$I = \sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right) + C$

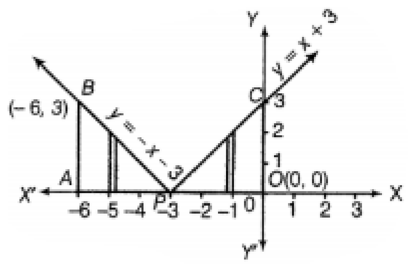
12. First, we sketch the graph of $y = |x + 3|$

$$\therefore y = |x+3| = \begin{cases} x+3, & \text{if} \quad x+3 \geq 0 \\ -(x+3), & \text{if} \quad x+3 < 0 \end{cases}$$

$$\Rightarrow y = |x+3| = \begin{cases} x+3, & \text{if} \quad x \geq -3 \\ -x-3, & \text{if} \quad x < -3 \end{cases}$$

So, we have $y = x + 3$ for x $\geq$ - 3 and y = -x - 3 for x < -3

A sketch of $y = |x + 3|$ is shown below:



$y=x+3$ is the straight line which cuts X and Y-axes at (-3, 0 and (0, 3), respectively.

$\therefore y=x+3$ for x $\geq$ -3 represents the part of the line which lies on the right side of x = -3.

Similarly,$y=-x-3$, x < -3 represents the part of line $y=-x-3$, which lies on left side of x = -3

Clearly, required area = Area of region ABPA + Area of region PCOP

$=\int_{-6}^{-3}(-x-3)dx+\int_{-3}^{0}(x+3)dx$

$=\left[-\frac{x^2}{2}-3x\right]_{-6}^{-3}+\left[\frac{x^2}{2}+3x\right]_{-3}^{0}$

$=\left[\left(-\frac{9}{2}+9\right)-(-18+18)\right]+\left[0-\left(\frac{9}{2}-9\right)\right]$

$=\left(-\frac{9}{2}-\frac{9}{2}\right)+(9+9)$

= 18 - 9

= 9 sq. units

OR

The given curves are,

y = 4 - x^2

$\Rightarrow$ x^2 = -(y - 4) ...(i)

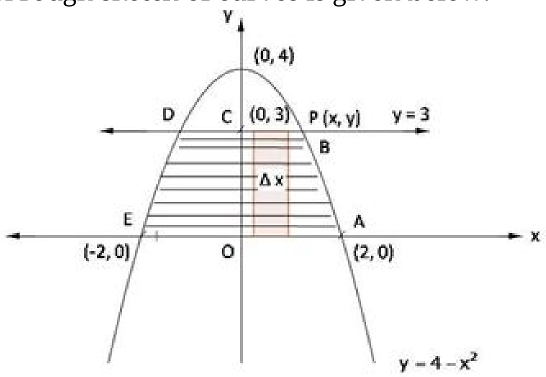
and y = 0 ...(ii)

y = 3 ...(iii)

Equation (i) represents a parabola with vertex (0, 4) and passes through (0, 2), (0, -2)

Equation (i) is x-axis and equation (iii) is a line parallel to x-axis passing through (0, 3)

A rough sketch of curves is given below:-



Shaded region represents the required area. We slice it in approximation rectangle with its Width $=\triangle x$ and length = y - 0 = y

Area of the rectangle $=y\triangle x$

This approximation rectangle can slide from x = 0 to x = 2 for region OABCI.

Therefore,we have,

Required area = Region ABDEA

= 2(Region OABCO)

$=2\int_0^2 y dx$

$=2\int_0^2\left(4-x^2\right)dx$

$=2\left(4x-\frac{x^3}{3}\right)_0^2$

$=2\left[\left(8-\frac{8}{3}\right)-(0)\right]$

Required area $=\frac{32}{3}$ square units

13. The equation of the plane in the intercept form is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Distance of this plane from the origin is given to be p.

$\therefore p=\frac{\left|\frac{1}{a}\times 0+\frac{1}{b}\times 0+\frac{1}{c}\times 0-1\right|}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}$

$\Rightarrow p=\frac{1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}$

$\Rightarrow\frac{1}{p}=\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}$

$\Rightarrow \frac{1}{p^2} = \left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2$
$\Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

CASE-BASED/DATA-BASED

14. Bag I: 3 red balls and 0 white ball.
Bag II: 2 red balls and 1 white ball.
Bag III: 0 red ball and 3 white balls.
Let E_1, $E_2$and E_3 be the events that bag I, bag II and bag III is selected and a ball is chosen from it.
$P(E_1) = \frac{1}{6}, P(E_2) = \frac{2}{6}$ and $P(E_3) = \frac{3}{6}$
i. Let E be the event that a red ball is selected. Then, Probability that red ball will be selected.
$P(E) = P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot \left(\frac{E}{E_2}\right) + P(E_3) \cdot P\left(\frac{E}{E_3}\right)$
$= \frac{1}{6} \cdot \frac{3}{3} + \frac{2}{6} \cdot \frac{2}{3} + \frac{3}{6} \cdot 0$
$= \frac{1}{6} + \frac{2}{9} + 0$
$= \frac{3+4}{18} = \frac{7}{18}$
ii. Let F be the event that a white ball is selected.
$\therefore P(F) = P(E_1) \cdot P\left(\frac{F}{E_1}\right) + P(E_2) \cdot P\left(\frac{F}{E_2}\right) + P(E_3) \cdot P\left(\frac{F}{E_3}\right)$
$= \frac{1}{6} \cdot 0 + \frac{2}{6} \cdot \frac{1}{3} + \frac{3}{6} \cdot 1 = \frac{1}{9} + \frac{3}{6} = \frac{11}{18}$
Note: P(F) = 1 - P(E) $= 1 - \frac{7}{18} = \frac{11}{18}$ [since, we know that P(E) + P(F) = 1]